Topology Ex. sheet 2

Benedikt Schmidt

1

let $X = \{a, b, c\}$

two possible topologies are: (see ex. sheet 1, Nr. 1)

$\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a,b\}\}$ neither one is a subset of the other

$\mathcal{T}_2 = \{X, \emptyset, \{a\}, \{b,c\}\}$ both contain 4 open sets.

so $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable ✓

4

Claim: Given 2 top. spaces X and Y, the top. spaces $X \times Y$ and $Y \times X$ are homeomorphic

proof: show that there is a homeomorphism $f: X \times Y \to Y \times X$

let $f(x,y) = (y,x)$

to show that f is a homeomorphism, we must show that f is bijective, continuous and f^{-1} is continuous

surjective: let $(z,w) \in Y$. Then $f(w,z) = (z,w)$

injective: let $(x,y), (z,w) \in X$ and suppose that $f(x,y) = f(z,w)$ then $(y,x) = (w,z)$ which implies $y = w$ and $x = z$, so $(x,y) = (z,w)$

you can also give f^{-1}

continuity: let $V \times U \subseteq Y \times X$ where V is open in Y and U is open in X

then: $f^{-1}(V \times U) = U \times V$

but U is open in X and V is open in Y, so $f^{-1}(V \times U)$ is open in $U \times V$

It is not true that every open is of the form $U \times V$

therefore f is continuous

continuity f^{-1}: let $U \times V \subseteq X \times Y$ where U is open in X and V is open in Y,

then: $f(U \times V) = V \times U$

but V is open in Y and U is open in X, so f^{-1} is continuous

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so $X \times Y$ and $Y \times X$ are homeomorphic $\square$

3 | $\mathcal{B} \subset 2^X$ be a cover of X closed under finite intersections

a subset of X is open if it is a union of sets from $\mathcal{B}$

then we get a topology on X whose Base is $\mathcal{B}$
$\overset{\shortparallel}{\mathcal{T}_B}$

claim: ~~if~~ a finite intersection of elements ~~from~~ in $\mathcal{T}_B$ is an element of $\mathcal{T}_B$

proof: we need to show: for any k: $\bigcap_{j=1}^{k} \left(\bigcup_{\alpha_j} B_{\alpha_j} \right) \in \mathcal{T}_B$

by induction

IV: $k=2$: $\bigcup_{\alpha_1} B_{\alpha_1} \cap \bigcup_{\alpha_2} B_{\alpha_2} = \bigcup_{\alpha_2} \left(B_{\alpha_2} \cap \bigcup_{\alpha_1} B_{\alpha_1} \right) = \bigcup_{\alpha_2} \bigcup_{\alpha_1} \underbrace{\left(B_{\alpha_2} \cap B_{\alpha_1} \right)}_{\in \mathcal{B} \text{ since closed under finite intersections}}$

IH: assume it holds for k

~~IS: $k \to k+1$ $\bigcup_{\alpha_n} B_{\alpha_n} \cap \bigcup_{\alpha_{n+1}} B_{\alpha_{n+1}}$~~

IS: $k \to k+1$

$$\bigcap_{j=1}^{k} \left(\bigcup_{\alpha_j} B_{\alpha_j} \right) \cap \bigcup_{\alpha_{j+1}} B_{\alpha_{j+1}} = \bigcup_{\alpha_{j+1}} \left(B_{\alpha_{j+1}} \cap \bigcap_{j=1}^{k} \left(\bigcup_{\alpha_j} B_{\alpha_j} \right) \right)$$

$$= \bigcup_{\alpha_{j+1}} \bigcup_{\alpha_j} \left(B_{\alpha_{j+1}} \cap \bigcap_{j=1}^{k} B_{\alpha_j} \right) = \bigcup_{\alpha_{j+1}} \bigcup_{\alpha_j} \underbrace{\left(\bigcap_{j=1}^{k+1} B_{\alpha_j} \right)}_{\in \mathcal{B} \text{ since closed under finite intersections}}$$

Where do you use induction hypothesis ?

2 | let $X = \{a, b, c\}$, then $\mathcal{T} = \{\mathcal{P}(X)\}$ is a topology

$\mathcal{P}(X)$ is a base of this topology since every set in $\mathcal{T}$ is a union of elements from $\mathcal{T}$.

now take set $\{a, b, c\} \in \mathcal{T}$

we have $\{a, b, c\} = \{a\} \cup \{b\} \cup \{c\}$ with $\{a\}, \{b\}, \{c\} \in \mathcal{P}(X)$

and $\{a, b, c\} = \{a, b\} \cup \{c\}$ with $\{a, b\}, \{c\} \in \mathcal{P}(X)$

so the decomposition by base is not unique

Topology Ex. sheet 3

Benedikt Schmidt

1) let (X,τ) be a topological space and $Y \subset X$.

(1) define $\tau_Y = \{U \cap Y : U \in \tau\}$ "subspace topology"

claim: (Y, τ_Y) is a topological space

proof: (i) we have $\emptyset, X \in \tau$ since (X,τ) is a topological space

therefore we have $\emptyset \cap Y = \emptyset \in \tau_Y$ and $X \cap Y = Y \in \tau_Y$

(ii) if $U_1, \dots, U_n \in \tau$, then $U_1 \cap \dots \cap U_n \in \tau$

so for $U_1 \cap Y, \dots, U_n \cap Y \in \tau_Y$ we have $U_1 \cap Y \cap U_2 \cap Y \cap \dots \cap U_n \cap Y$

$= (U_1 \cap \dots \cap U_n \cap Y) \in \tau_Y$

$\bigcap_{i=1}^{n} V_i = \bigcap_{i=1}^{n} (U_i \cap Y) = \left(\bigcap_{i=1}^{n} U_i\right) \cap Y \in \tau_Y$

(iii) let Λ be any indexing set and $U_\lambda \in \tau \;\; \forall \lambda \in \Lambda$

then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau$

so $\forall \lambda \in \Lambda$ we have $(U_\lambda \cap Y) \in \tau_Y$

now $\bigcup_{\lambda \in \Lambda} (U_\lambda \cap Y) = \underbrace{\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right)}_{\in \tau} \cap Y \in \tau_Y$

(ii)

claim: if $\mathcal{B}$ is a base of τ, then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a base of τ_Y

proof: if $\mathcal{B}$ is a base of τ, then every open set in X is a union of sets from $\mathcal{B}$

let $\mathcal{S}$ be a union of sets from $\mathcal{B}$ s.t. $X = \bigcup_{S \in \mathcal{S}} S$

then: $Y = X \cap Y$

$= \left(\bigcup_{S \in \mathcal{S}} S\right) \cap Y$

$= \bigcup_{S \in \mathcal{S}} (S \cap Y)$ but $S \in \mathcal{B}$, so $S \cap H \in \mathcal{B}_Y$

so Y is a union of sets from $\mathcal{B}_Y$

2| let Y be a subspace of X

claim: a set A is closed in Y $\iff$ A equals the intersection of a closed set of X with Y

proof:

"$\Rightarrow$" assume that A is closed in Y

then $Y \setminus A$ is open in Y, so by definition it equals the intersection of an open set U of X with Y: $Y \setminus A = U \cap Y$

$X \setminus U$ is closed in X and $A = Y \cap (X \setminus U)$

so A equals the intersection of a closed set of X with Y

"$\Leftarrow$" assume $A = C \cap Y$ where C is closed in X

then $X \setminus C$ is open in X

so $(X \setminus C) \cap Y$ is open in Y by definition of subspace

but $(X \setminus C) \cap Y = Y \setminus A$ which is open

so A is closed in Y

3| let Y be a subspace of X and $A \subset Y$

claim: $Cl_Y A = (Cl_X A) \cap Y$ hier muss geklammert werden

proof:

"$\subseteq$" $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is close in Y by ex. 2

we have $A \subset A \cap Y$ and $Cl_Y A$ is intersection of all closed subsets of Y containing A

so $Cl_Y A \subset \overset{Cl_X A}{\bar{A}} \cap Y$

"$\supseteq$" $Cl_Y A$ is closed in Y

so, by ex. 2, $Cl_Y A = C \cap Y$ for some set C closed in X

then C contains A

$\bar{A}$ is the intersection of all such closed sets, so $\bar{A} \subset C$

then $(\bar{A} \cap Y) \subset (C \cap Y) = Cl_Y A$

4) claim: $x \in \mathrm{Cl} A \Leftrightarrow$ every set from $\mathcal{B}$ containing x intersects A (*)
where the topology on X is given by a base $\mathcal{B}$

proof: we proof a slightly more general equivalence : (see lecture)

$x \notin \mathrm{Cl} A \Leftrightarrow \exists$ open set $U \ni x$ that does not intersect A (**)

"$\Rightarrow$" if $x \notin \mathrm{Cl} A$ then $U = X \setminus \bar{A}$ is an open set containing x that does not intersect A

"$\Leftarrow$": if there exists open set U containing x which does not intersect A, then $X \setminus U$ is a closed set containing A
x cannot be in $\mathrm{Cl} A$ then

now back to our originally tackled claim:

if every open set containing x intersects A, so does every $B \in \mathcal{B}$

if every $B \in \mathcal{B}$ containing x intersects A, so does every open set U containing x, because U contains a basis element that contains x

so (*) and (**) are equivalent □

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5) consider $\mathbb{R}$ with standard topology

$A = \{ [1,2] \cap \mathbb{Q} \} \cup \{3,4\} \subset \mathbb{R}$

intA is empty

$\partial A = \{ [1,2] \} \cup \{3,4\}$ since $\partial \mathbb{Q} = \mathbb{R}$ and every neighborhood of 3 and 4 intersects A^c

$\mathrm{Ext} A =$ interior of its complement
$= \mathbb{R} \setminus \{ [1,2] \cup \{3,4\} \}$

$\mathrm{Cl} A = A \cup \partial A = \{ [1,2] \} \cup \{3,4\}$

$A_\ell = \{ [1,2] \}$ since every neighborhood of 1 and 2 contains rational numbers and $B_{0.5}(3)$ or $B_{0.5}(4)$ does not contain elements of A

Topology ex 4

1

<u>claim:</u> in a Hausdorff space, a sequence of points converges to at most one point

<u>proof:</u> a sequence is convergent, $x_n \to x$, if there is an element $x \in X$ s.t. for every open neighborhood U of x, there exists n_0 s.t. $n > n_0$ implies $x_n \in U$

let X Hausdorff and $x, y \in X$ s.t. $x \neq y$

then $\exists$ disjoint open sets U, V st. $x \in U$ and $y \in V$

if $x_n \to x$, then all but a finite number of elements of x_n lies in U

then only a finite number of can be in V, so it contradicts $x_n \to y$

2

<u>claim:</u> the product of two Hausdorff spaces is Hausdorff

<u>proof:</u> let X, Y be Hausdorff spaces and $(x,y), (z,w) \in X \times Y$ distinct

Suppose $x \neq z$

then $\exists$ disjoint open sets $U, V \subset X$ s.t. $x \in U$ and $z \in V$

hence $(x,y) \in U \times Y$ and $(z,w) \in V \times Y$

$$U \times Y \cap V \times Y = \underbrace{(U \cap V)}_{=\emptyset} \times Y = \emptyset$$

5

Let $f: X \to Y$ be a cont. map of top. spaces and Y be Hausdorff.

Let $\Gamma = \{(x,y) \in X \times Y : y = f(x)\}$ the graph of f

claim: Γ is a closed set

proof: let $(x,y) \in (X \times Y) \setminus \Gamma$

then $y \neq f(x)$

since Y is Hausdorff, $\exists$ disjoint open subsets U, V in Y s.t. $y \in U$ and $f(x) \in V$

since f is cont., $\exists$ open neighborhood W of x s.t. $f(W) \subseteq V$

clearly, $W \times V$ is an open neighborhood of (x,y) and $W \times U$ is disjoint from Γ

so $(X \times Y) \setminus \Gamma$ is open, hence Γ is closed

3

Consider $\mathbb{R}$ with cofinite topology, i.e. $C \subset \mathbb{R}$ is closed iff $C = \mathbb{R}$ or C is finite

let $x_n = \frac{1}{n}$

def: A sequence $\{x_n\}_n$ of a top. space X converges to $x \in X$ if for every neighborhood $U \ni x$ $\exists$ $N \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq N$

limit: we have $x_n \in [0,1]$

let $x \in [0,1]$ and U a neighborhood of x

then U is infinite since we are in cofinite topology

hence $\exists$ $N \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq N$

Therefore, every $x \in [0,1]$ is a limit point of x_n

now consider the point O

every neighborhood V of O contains infinitely many points

therefore $\exists N \in \mathbb{N}$ s.t. $x_n \in V \quad \forall n \geq N$

4

T_3: (regular Hausdorff) All points are closed and for every closed set $Z \subset X$ and every $x \notin Z$ there are neighborhoods of Z and x which are disjoint

$T_{2\frac{1}{2}}$: (Urysohn) Every two points have nghb whose closures are disjoint

claim: $T_3 \Rightarrow T_{2\frac{1}{2}}$

proof: assume that X is regular and satisfies T_1

let $x \in X$ and $y \in X$

Why can you take?

let $x \in Z$ where Z is a closed set in X and $y \notin Z$

then, according to T_3, we have disjoint nbgh. $U \supset Z$ and $V \ni y$

let $W \subseteq V$ closed s.t. $y \in W$

W can be $\{y\}$?

we have that $\overline{U}$ and W are disjoint

now we can again apply regularity condition to get $\hat{U}, \hat{V}$ with $\hat{U} \supset \overline{U}$ and $\hat{V} \supset W$

then we have $\overline{\hat{V}}$ and $\overline{U}$ are disjoint

Why?

but then, $\hat{V}$ and U are open sets containing W and Z with disjoint closures

Topology exercise sheet 5

1|

$\{X_\lambda\}$ is a family of topological spaces, $X = \prod_{\lambda\in\Lambda} X_\lambda$ is a product space with Tychanoff topology

claim: all X_λ are Hausdorff $\Rightarrow$ X is Hausdorff

proof: let $x, y \in X$ s.t. $x \neq y$

then there is at least one X_{λ_0} that x and y differ on

hence $x_{\lambda_0} \neq y_{\lambda_0}$

since X_{λ_0} is Hausdorff, there exist open sets U_{λ_0} and V_{λ_0} in X_{λ_0}, s.t.

$x_{\lambda_0} \subseteq U_{\lambda_0}$ and $y_{\alpha_0} \subseteq V_{\lambda_0}$ and $U_{\lambda_0} \cap V_{\lambda_0} = \emptyset$

now let $U_\lambda = V_\lambda = X_\lambda \ \forall \lambda \in \Lambda \setminus \{\lambda_0\}$

let $U = \prod_{\lambda\in\Lambda} U_\lambda$ and $V = \prod_{\lambda\in\Lambda} V_\lambda$

then U and V are open sets in X and $x \in U$ and $y \in V$

now I claim that $U \cap V = \emptyset$

assume $z \in U \cap V$, then $z_{\lambda_0} \in U_{\lambda_0} \cap V_{\lambda_0} = \emptyset$

thus X is Hausdorff

2|

claim: a sequence of points $\{x_n\}_{n=1}^{\infty}$ in X converges $\Leftrightarrow$ the sequence $\{\pi_\lambda(x_n)\}_{n=1}^{\infty}$ converges to $\pi_\lambda(x)$ $\forall \lambda \in \Lambda$

$\Rightarrow$ let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X, s.t $x_n \to x$

since π_λ is continuous, we have $\pi_\lambda(x_n) \to \pi_\lambda(x)$ $\forall \lambda \in \Lambda$

$\Leftarrow$ we must check that for any subbasis set S on the product topology, $\exists\, N \in \mathbb{N}$ s.t. $x_n \in S \;\forall\, n \geq N$

we have $S = \pi_\lambda^{-1}(U_\lambda)$ for some $\lambda \in \Lambda$ and some open set $U_\lambda \subset X_\lambda$

now since $\pi_\lambda(x_n) \to \pi_\lambda(x)$, $\exists\, N \in \mathbb{N}$ s.t. $\pi_\lambda(x_n) \in U_\lambda \;\forall\, n \geq N$

but for those n we have $x_n \in \pi_\lambda^{-1}(U_\lambda) = S$

so we prooted: if $x \in V$, for an open set V, then $x \in U$, for an open basic set U

(because every open set is a union of basic open sets)

$U = U_1 \cap \dots \cap U_k$ for some subbasic sets $U_1, \dots, U_k$

so $x \in U_i$ for $1 \leq i \leq k$

if $x_n \in U_i$ whenever $n \geq N_i$, then $x_n \in U$ for $n \geq \max\{N_1, \dots, N_k\}$

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3 let $A_\lambda \in X_\lambda$, $\overline{A_\lambda} :=$ closure of A_λ in X_λ

claim: $\prod_{\lambda\in\Lambda} \overline{A_\lambda} = \overline{\prod_{\lambda\in\Lambda} A_\lambda}$

"$\subseteq$": suppose that $x = (x_1, \dots, x_n) \in \prod_{\lambda\in\Lambda} \overline{A_\lambda}$

indexing set muss nicht endlich sein

Λ may not be finite. $(x = (x_\lambda)_{\lambda\in\Lambda})$

then $x_\lambda \in \overline{A_\lambda}$ $\forall\, \lambda \in \{1, \dots, n\}$

let $U = \prod_{\lambda\in\Lambda} U_\lambda$ be an open base neighborhood of x in $\prod_{\lambda\in\Lambda} X_\lambda$

then, U_λ is an open neighborhood of x_λ for each $\lambda \in \Lambda$

hence: $A_\lambda \cap U_\lambda \neq \emptyset$

therefore: $\prod_{\lambda\in\Lambda} A_\lambda \cap U_\lambda = \left(\prod_{\lambda\in\Lambda} A_\lambda\right) \cap \left(\prod_{\lambda\in\Lambda} U_\lambda\right) = \left(\prod_{\lambda\in\Lambda} A_\lambda\right) \cap U \neq \emptyset$

now we can apply Proposition 1, part (1) to conclude that

$x \in \overline{\prod_{\lambda \in \Lambda} A_\lambda}$

"$\supseteq$" let $x = (x_1, \dots, x_n) \in \prod_{\lambda \in \Lambda} \overline{A_\lambda}$. For each $\lambda \in \Lambda$, let U_λ be an open neighborhood of x_λ

then $U = \prod_{\lambda \in \Lambda} U_\lambda$ is an open nghb. of x.

NO IN GENERAL

hence: $\left(\prod_{\lambda \in \Lambda} A_\lambda\right) \cap U = \left(\prod_{\lambda \in \Lambda} A_\lambda\right) \cap \left(\prod_{\lambda \in \Lambda} U_\lambda\right) = \prod_{\lambda \in \Lambda} A_\lambda \cap U_\lambda \neq \emptyset$

$\Rightarrow A_\lambda \cap U_\lambda \neq \emptyset \quad \forall \lambda \in \Lambda$

$\overset{\text{Prop. 1}}{\Longrightarrow} x_\lambda \in \overline{A_\lambda} \quad \forall \lambda \in \Lambda$

and hence $x \in \prod_{\lambda \in \Lambda} \overline{A_\lambda}$

4 consider the half-plane $H = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$

define the topology on H to be generated by the following sets:

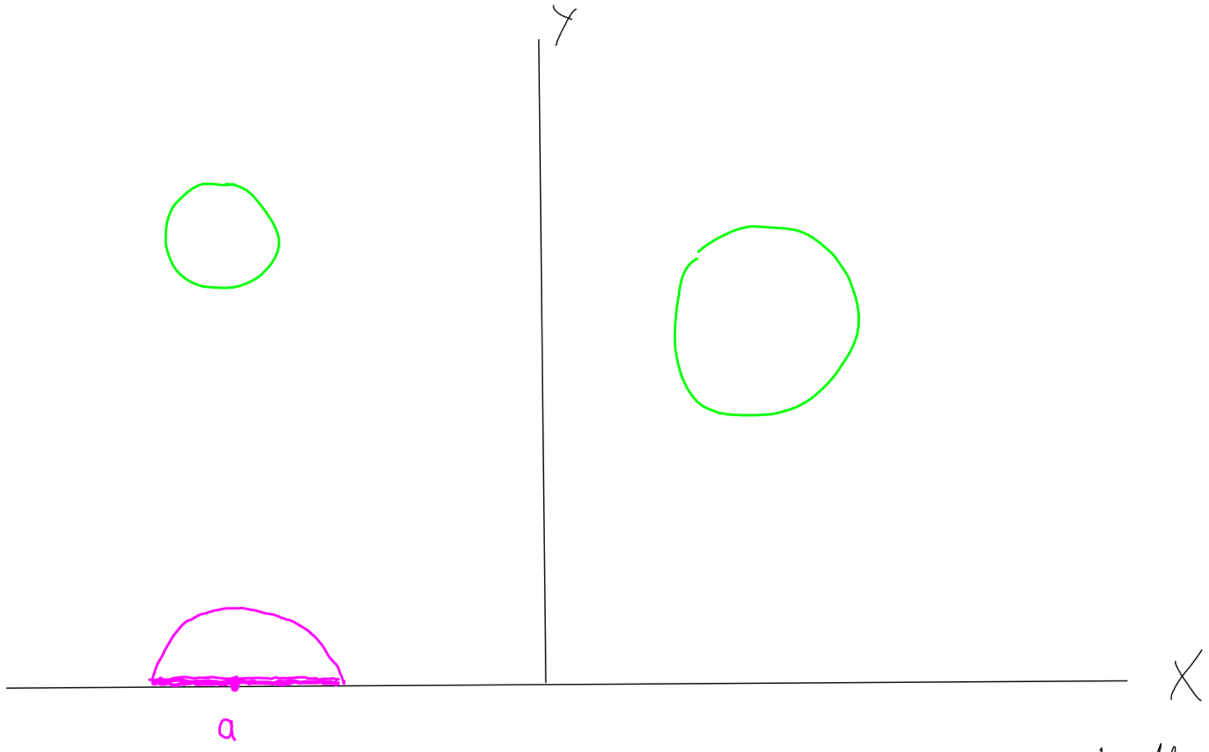
•) usual open balls contained in H

•) sets of the form: $H \cap B_r((a,0)) \setminus \{(x,0) \in \mathbb{R}^2 : x \in (a-r, a+r) \setminus \{a\}\}$

claim: this topology is Hausdorff

proof: we have "two kinds" of open sets

therefore, we have 3 possibilities for the position of a pair of points



possibility 1 both points of the pair are contained in an open ball
then we can take open sets U, V s.t. $P_1 \in U$ and $P_2 \in V$ and $U \cap V = \emptyset$

possibility 2: both points of the pair are contained in the second type of open set

possibility 3 P_1 is contained in a usual open ball and P_2 in an open set of the 2nd type

Topology ex. sheet 6

1 Sorgenfrey line

consider $\mathbb{R}$ and $\mathcal{B} = \{[a,b)\}$ the collection of all half-open intervals

this the lower limit topology on $\mathbb{R}$

$\mathbb{R}_\ell$ is the resulting topological space

(1)

<u>claim:</u> $\mathbb{R}_\ell$ is not second countable

<u>proof:</u> A topol. space is called second countable if it has a countable base.

assume that B is a countable basis of $\mathbb{R}_\ell$

given a point x, let $B_x \in B$ be a basis element such

that $x \in B_x \subseteq [x, x+1)$

then, if $x \neq y$, we have $B_x \neq B_y$ since $x = \inf B_x \neq \inf B_y = y$

so there is an injective function $f: \mathbb{R}_\ell \to B$
$x \mapsto B_x$

hence B cannot be countable

(2)

<u>claim:</u> $\mathbb{R}_\ell$ is normal

<u>proof:</u> A space X is called normal, if for every two disjoint closed sets A and B there exist disjoint open neighborhoods of A and B

take any two disjoint closed sets A and B in $\mathbb{R}_\ell$

for each $a \in A$, choose $a' > a$ s.t. $\underbrace{[a,a')}_{\text{open}} \cap B = \emptyset$

then $U = \bigcup_{a \in A} [a,a')$ is an open set containing A

for each $b \in B$, choose $b' > b$ s.t. $[b,b') \cap A = \emptyset$

then $V = \bigcup_{b \in B} [b,b')$ is an open set containing B

to show that $U \cap V = \emptyset$, it is enough to show that

$$[a,a') \cap [b,b') = \emptyset \quad \forall a \in A, b \in B$$

suppose that $a \in A$, $b \in B$, wlog $a < b$

since $[a,a') \cap B = \emptyset$ it follows that $b \geq a'$

hence $[a,a') \cap [b,b') = \emptyset$

$\boxed{4}$

claim: a Tychonoff product of regular spaces is regular

proof: let $\{X_i\}_{i \in I}$ be a family of regular spaces and $X = \prod_{i \in I} X_i$

we have to show that, given any point $x \in X$ and any open subset U of X containing x, there is an open subset $\overline{V} \ni x$ s.t $\overline{V} \in U$

there exists a basis element B of X containing x s.t. $B \subset U$

Write $B = \prod_{i \in I} B_i$

for those B_i that are equal to the whole space, define $V_i = B_i$.

for those B_i that are proper subsets of X_i, we can use the fact that X_i is regular: with remark 1 on the ex. sheet, we can find an open set $V_i \ni x_i$ in X_i s.t. $\overline{V_i} \subset B_i$

now consider $V := \prod_{i \in I} V_i$

we have $x \in V$ by construction and V is a basis element for the topology on X, so V is open

we have $\overline{V} \subset B \subset U$

$\subset$ How do you prove it?

hence X is regular

$\boxed{2}$ let X be a top. space satisfying (T_1).

claim: X is normal $\iff$ for every closed set $A \subset X$ and its neighborhood O_A there exists a nghb. O_A' with $\overline{O_A'} \subset O_A$

"$\Rightarrow$" let X be normal, i.e. for every two disjoint closed sets A and B, there exist disjoint open nghb. of A and B

let $A \subset X$ closed and $U \in \mathcal{T}$, s.t. $A \subset U$

then $X \setminus U$ is closed and $A \cap X \setminus U = \emptyset$

since X is (T_4), there exist disjoint open sets $O_A', V_2 \in \mathcal{T}$ s.t. $A \subseteq O_A'$

and $X \setminus U \subset V_2$

hence $O_A' \subset X \setminus V_2$ and $X \setminus V_2$ is closed

thus $\overline{O_A'} \subset \underbrace{X \setminus V_2}_{:O_A} \subset X \setminus (X \setminus U) = U$

"$\Leftarrow$" let $A, B \subset X$ closed sets and $A \cap B = \emptyset$

we have to show that there exist disjoint

open neighb. of A and B

we know that if O_A is a neighb. of A, there Exists O_A'

s.t. $\overline{O_A'} \in O_A$

then $X \setminus B$ is open and $A \subset X \setminus B$

hence there is $V \in \mathcal{T}$ s.t. $A \subset V \subset \overline{V} \subset X \setminus B$

we choose $U_A := V$ and $U_B := X \setminus \overline{V}$

then $A \subset U_A$, $B \subseteq U_B$ since $\overline{V} \subseteq X \setminus B$

and $U_A \cap U_B = \emptyset$ since $V \subset \overline{V}$

3

if X is a normal top. space and U_0, U_1 are two open sets with $U_0 \Subset U_1$, then there exists an open set $U_{1/2}$ s.t.

$$U_0 \Subset U_{\frac{1}{2}} \Subset U_1$$

proof: let $U_0 \Subset U_1$

then $\bar{U}_0 \subset U_1$, hence U_1 is an open neighbourhood of $\bar{U}_0$

now we can apply ex. 2 to find $U_{\frac{1}{2}}$, s.t.

$$U_0 \Subset U_{\frac{1}{2}} \Subset U_1$$

Topology ex. sheet 7

1|

let X be a compact space and $A \subset X$ a closed subset

claim: A is compact in X

proof: let $A \subset X$ be closed

let $\{U_\alpha\}$ be an open cover of A

then $\{U_\alpha\}_{\alpha \in \mathbb{N}} \cup \{A^c\}$ is a cover of X

since X is compact, we can extract a finite subcover

$\{U_{\alpha_1}, \dots, U_{\alpha_n}, A^c\}$ from $\{U_\alpha\}_{\alpha \in \mathbb{N}} \cup \{A^c\}$

since $A \cap A^c = \emptyset$ we have that $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover of A $\square$

2|

Consider the cofinite top. on $\mathbb{R}$

claim: compact $\not\Rightarrow$ closed

proof: take a singleton $\{x\} \subset \mathbb{R}$

$\{x\}$ is closed!!

note: A is closed $\Leftrightarrow$ $\# A$ is finite.

let $\{U_i\}_{i \in I}$ be a cover of $\{x\}$

then $\exists\, i_0 \in I$ s.t. $\{x\} \subset U_{i_0}$, hence U_{i_0} is a finite subcover of $\{x\}$

thus $\{x\}$ is compact

but: $\{x\}$ is not closed, otherwise $\{x\}^c$ would have to be finite which is not case $\square$

3 | let X be a set with cofinite topology

claim: X is compact

proof: cofinite topology: $\mathcal{T} = \{A \subseteq X \mid A = \emptyset$ or $X \setminus A$ is finite$\}$

let $\{U_i\}_{i \in I}$ be an open cover X

then $X \setminus U$ is finite for some $U \in \{U_i\}_{i \in I}$

for every $a \in X \setminus U$, let U_a be an element of $\{U_i\}_{i \in I}$ that contains a

then $\{U\} \cup \{U_a : a \in X \setminus U\} \subseteq \{U_i\}_{i \in I}$

is a finite subcover of X

claim: every subset of X is compact

proof: let $A \subset X$ and $\{U_\alpha\}_{\alpha \in A}$ be an open cover of A

for every $U_{\alpha^*} \subseteq \{U_\alpha\}_{\alpha \in A}$, $U_{\alpha^*}^c$ has finitely many elements, since U_{α^*} is open

hence, there are only $\underbrace{\text{finitely many elements of } A}_{\{x_1, \dots, x_n\}}$, that are not in U_α

for each of those elements, find a $U_{\alpha_i} \ni x_i$

then $U_{\alpha^*} \cup \{U_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of A

ex. 2 (any open set other than $\emptyset$ and X is compact, but not closed)

4

claim: Y is closed

proof:

Let X be a Hausdorff space and $Y \subset X$ compact.

we show that $X \setminus Y$ is open

let $x \in X \setminus Y$ and $y \in Y$

since X is Hausdorff, there exist open neighbarhoods U_x of and U_y of y s.t. $U_x \cap U_y = \emptyset$

then $\{U_y : y \in Y\}$ is an open cover of Y

since Y is compact, there is a finite subcover of Y

$\{U_y : y \in \{1, \dots, n\}\} \subseteq \{U_y : y \in Y\}$

but $\bigcap_{i=1}^{n} U_{x_i}$ is an open neighbarhood of x

and $\bigcap_{i=1}^{n} U_{x_i} \cap Y = \emptyset$

for every $x \in X \setminus Y$, we can find such a neighbarhood,

hence $X \setminus Y$ is open

thus Y is closed $\square$

5) (1)

claim: Y is compact $\Rightarrow \pi : X \times Y \to X$ is closed

proof: A map $f: X \to Y$ is called closed if it maps closed sets to closed sets

hence we need to show: for $C \subset X \times Y$ closed, $\pi(C) \subset X$ is closed

let $C \subset X \times Y$ closed

~~by Ex. 1, C is compact~~

$X \times Y$ is compact?.

~~since the projection π is continuous, the image of C is compact~~

by ex. 4, C as a compact subset of a Hausdorff space is closed $\square$

$X \times Y$ is Hausdorff space?.

(2) consider $p_1 : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to x$

let $A = \{(x, \frac{1}{x}) : x \neq 0\}$, then A is closed in $\mathbb{R}^2$

but: $p_1(A) = \mathbb{R} \setminus \{0\}$ which is not closed in $\mathbb{R}$

Topology ex. sheet 8

1|

claim: A is clopen $\Longleftrightarrow$ $\partial A = \emptyset$

proof: A is open iff $\partial A \cap A = \emptyset$

A is closed iff $\partial A \subseteq A$.

hence $\partial A = \emptyset$

How about the other direction?

2|

Let X be a top. space and $A \subset X$

claim: A is connected $\Longrightarrow$ $Cl(A)$ is connected

proof: suppose $Cl(A)$ is disconnected

then $\exists$ two non-empty open sets $H, K \subset Cl(A)$

s.t. $Cl(A) = H \cup K$ and $H \cap K = \emptyset$

H and K are open in $Cl(A)$ and $A \subset Cl(A)$

hence $A = \underbrace{(H \cap A)} \cup \underbrace{(K \cap A)}$

those are non-empty disjoint open sets in A

How to prove?

thus A is disconnected, a contradiction

3

$\mathcal{B} = \{[a,b)\}$, $\mathbb{R}_\ell := $ topology generated by $\mathcal{B}$ (aka Sorgenfrey line

claim: the Sorgenfrey line $\mathbb{R}_\ell$ is not connected

proof: by definition, a connected space has <u>no</u> non-empty proper clopen subsets

now take $I = [a,b) \in \mathcal{B}$, per definition, I is open

but $I^c = \mathbb{R}_\ell \setminus [a,b) = (-\infty, a) \cup [b, \infty)$

$$= \bigcup_{n=1}^{\infty} \underbrace{[-n, a)}_{\text{open}} \cup \underbrace{[b, \infty)}_{\text{open}}$$

hence I^c is open $\Rightarrow I$ is clopen ↯

4

let X and Y be two homeomorphic spaces

let C_X, C_Y be the sets of connected components of X and Y

claim: there is a bijection between C_X and C_Y

proof: a homeomorphism $f: X \to Y$ is continuous, bijective mapping, s.t. $f^{-1}: Y \to X$ is continuous as well.

let X_0 be a connected component of X

we have to show that X_0 is mapped into a single component of Y

assume the contrary: $f(X_0)$ intersects $Y_1, Y_2, \dots, Y_k$

then, $X_0 = \underbrace{\left(f^{-1}(Y_1) \cap X_0\right)}_{\text{open in } X_0} \cup \left(f^{-1}(Y_2) \cap X_0\right) \cup \dots \cup \left(f^{-1}(Y_k) \cap X_0\right)$

since all Y_i are disjoint, we get that X_0 is not connected ↯

we can apply the same argument for $f^{-1}: Y \to X$

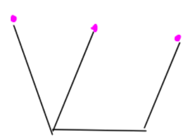
hence, each component of Y is mapped to a single component of X under f^{-1}

hence f is injective and surjective, thus bijective, since f is a homeomorph.

---

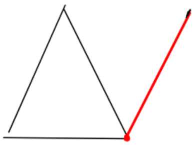
5 If two spaces have different number of cut-points, they are not homeomorphic

(a) has no cut-points since $X \setminus x$ is connected for every $x \in X$

(b) 

every point is a cut point except the points at the "tips"

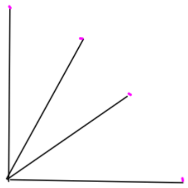
(c)



the last point is not a cut point

cut points in red

(d)



every point is a cut point except the points at the "tips"

hence (a) cannot be hom. to any other graphs since it is the only one without cut point

(c) has clearly fewer cut points than (b) and (d)

(b) has more cut points than (d) ?

In (b) & (d), there are infinitely many cut points, and we can't compare!

Topology ex. sheet 9

1)

claim: $\mathbb{R}/\mathbb{Q}$ is not Hausdorff

proof: we show that the quotient topology on $\mathbb{R}/\mathbb{Q}$ is the trivial topology, which is not Hausdorff

suppose that U is an open non-empty set in the quot. top. $\mathbb{R}/\mathbb{Q}$

let $\pi : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ be the canonical projection

let $[x] \in U$, then $\pi^{-1}(U)$ is an open set in $\mathbb{R}$ containing x

hence, there is an open interval (a,b) in $\pi^{-1}(U)$

for every $x \in \pi^{-1}(U)$ and $r \in \mathbb{Q}$, we have $x+r \in \pi^{-1}(U)$

since $\pi(x+r) = \pi(x)$

thus $\pi^{-1}(U)$ contains $\bigcup_{r\in\mathbb{Q}} (a+r, b+r)$

hence $\pi^{-1}(U) = \mathbb{R}$ and $U = \pi(\pi^{-1}(U)) = \pi(\mathbb{R}) = \mathbb{R}/\mathbb{Q}$

therefore $\mathbb{R}/\mathbb{Q}$ is not Hausdorff

2)

claim: $\mathbb{R}P^1$ is homeomorphic to S^1

proof: consider the map $f: S^1 \to S^1$
$z \mapsto z^2$

f is surjective and $f(z) = f(-z)$

let $\bar{f}: S^1 / \{\pm 1\} \cong \mathbb{R}P^1 \to S^1$

then $\bar{f}$ is a homeomorphism because it is continuous

and, since f is open, $\bar{f}$ is open as well

Why?

!

3)

claim: $C(S^1)$ is homeomorphic to $\mathbb{B}^2$

proof: consider the equivalence relation $(x,1) \sim (y,1)$ $\forall x,y \in S^1$

$C(S^1) = (S^1 \times [0,1]) / \sim$

let $f(\theta, t) = (1-t)\, e^{i\theta}$ s.t. $f: S^1 \times [0,1] \to \mathbb{B}^2$

we have, if $x \sim y$, $f(x,1) = 0 = f(y,1)$

hence f is constant on equivalence classes

by UMP, there is a unique cont. map $\bar{f}: (S^1 \times [0,1])/\sim \to B^2$

$\bar{f}([\theta, t]) = f(\theta, t)$ $\qquad f(\theta,t) = (1-t)\, e^{i\theta}$

claim: $\bar{f}$ is bijective

proof: f is a composition of bijective functions, hence $\bar{f}$ is bijective

$\bar{f}$ is continuous as a composition of cont. functions

if we show that f is open (or closed), we get that f is a homeomorphism but I don't know how

It is from $S^1 \times [0,1]/\sim$ is Compact & Hausdorff.

4)

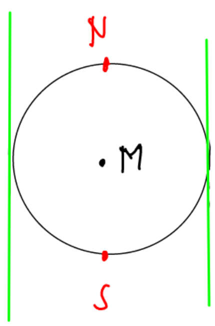
let $N = (0,0,1)$ and $S = (0,0,-1)$ be the north and the South poles of S^2

claim: $S^2 \setminus \{N,S\} \cong S^1 \times \mathbb{R}$

infinite cylinder

Define the map. (Mathematically!)

proof:



the cylinder touches the sphere along its equator

project every point in $S^2 \setminus \{N,S\}$ onto the cylinder along a ray through the centre of the sphere

This is a homeomorphism

5|

claim: $\Sigma(S^1) \cong S^2$

proof: we have $\Sigma(S^1) = (S \times [-1,1])/\sim$

where $(x,1) \sim (y,1)$ and $(x,-1) \sim (y,-1)$ $\forall\ x,y \in S^1$

let $f: S^1 \times [\cancel{0},1] \to S^2$ -1

$((x,y),t) \mapsto (x\sqrt{1-t^2}, y\sqrt{1-t^2}, t)$

f is constant on equivalence classes since:

let $((x,y),1) \sim ((a,b),1)$

then $f((x,y),1) = (0,0,1) = f((a,b),1)$

similar for $((x,y),-1) \sim ((a,b),-1)$

hence f is constant on equivalence classes

by UMP we get $\bar{f}: (S^1 \times [-1,1])/\sim \to S^2$

such that the diagramm commutes

we have $\bar{f}([(x,y),t]) = f((x,y),t)$

f is cont. as a composition of cont. functions

claim: $\bar{f}$ is surjective

proof: $f: ((x,y),t) \mapsto (x\sqrt{1-t^2}, y\sqrt{1-t^2}, t)$

the only problem is $t=1$ or $t=-1$, but this is solved by the equivalence classes

claim: $\bar{f}$ is injective

proof: let $\bar{f}([(a,b),c]) = \bar{f}([(x,y),z])$

hence $f((a,b),c) = f((x,y),z)$

$\Rightarrow \sqrt{a\sqrt{1-t^2}, b\sqrt{1-t^2}, c} = \sqrt{x\sqrt{1-t^2}, y\sqrt{1-t^2}, z}$

hence $c=z$ and $a=x$ and $b=y$

thus f is bijective

claim: $\bar{f}$ is a homeomorphism

proof: we can show that f is open to proof the claim

take an open ball centered at $(0,0,1)$

then $f^{-1}(U)$ is $S^1 \times (1-\varepsilon, 1]$ which is open

same for $(0,0,-1)$

Topology ex. sheet 10

1)

claim: $\mathbb{Q}$ is not locally compact

proof: I assume standard topology

let $x \in \mathbb{Q}$, $N \subseteq \mathbb{Q}$ a neighborhood of x

then $\exists U \in \mathcal{T} : x \in U \subseteq N \subseteq \mathbb{Q}$

now suppose that N is compact:

then, N is nowhere dense

why?

thus $\overline{N}$ contains no open set of $\mathbb{Q}$ which is non-empty.

but U is a non-empty open set of $\mathbb{Q}$:

$N \subseteq \overline{N}$, hence $U \subseteq \overline{N}$ $\lightning$

2)

let Y be a one-point compactification of X.

claim: X is locally compact

proof: let $x \in X$

pick open subsets U, V of Y s.t. $x \in U$, $\infty \in V$

and $U \cap V = \emptyset$

then $V = Y \setminus C$ for some closed compact subset C of X

then $x \in U \subseteq C$

since $Cl_X(U)$ is closed in C, it is compact

3

claim: $\mathbb{R}^{\mathbb{N}}$ is not locally compact

proof: note that the projection from $\mathbb{R}^{\mathbb{N}}$ to any factor,

say the k^{th} factor, is continuous.

for if (a_k, b_k) is a basic open set in the k^{th} factor,

then $\pi^{-1}((a_k, b_k)) = \mathbb{R} \times \dots \times \mathbb{R} \times (a_k, b_k) \times \mathbb{R} \times \mathbb{R} \times \dots$

$$= \bigcup_{j=1}^{\infty} (-j, j) \times \dots \times (-j, j) \times (a_k, b_k) \times \mathbb{R} \times \mathbb{R} \times \dots$$

which is a union of basic open sets.

now suppose that $\mathbb{R}^{\mathbb{N}}$ is locally compact

Then $\exists$ a compact set C that contains an open neighborhood of $(0,0,\dots)$, hence contains a basic open set $U=(a_1,b_1)\times\dots\times(a_n,b_n)\times\mathbb{R}\times\mathbb{R}\times\dots$

for every $x\in\mathbb{R}$, the $(n+1)$-st factor

$$(0,\dots,0,x,0,\dots)\in U\subseteq C \quad \text{and}$$

$$\pi_{n+1}((0,\dots,0,x,0,\dots))=x, \quad \text{so } \pi_{n+1}:C\to\mathbb{R} \text{ is surjective}$$

but, since $\mathbb{R}$ is not compact, this is a contradiction

Topology ex. sheet 11

1

let M_1 and M_2 be two topol. manifolds of dimensions n and k

claim: $M_1 \times M_2$ is homeomorphic to $\mathbb{R}^{n+k}$ **no!**

proof: every point $(x,y) \in M_1 \times M_2$ has a product open set $U \times V$

where $U \subseteq M_1$, $V \subseteq M_2$ are both open with $x \in U$, $y \in V$

You should start with this!

we can pick U and V s.t. $U \cong \mathbb{R}^n$ and $V \cong \mathbb{R}^k$

there are homeomorphisms $f: U \to \mathbb{R}^n$, $g: V \to \mathbb{R}^k$

we define $F: U \times V \to \mathbb{R}^n \times \mathbb{R}^k$

$$(a,b) \mapsto (f(a), g(b))$$

continuity: F is continuous, since its components are cont.

inverse: F^{-1} exists since f^{-1} and g^{-1} exist

F^{-1} is cont. since f^{-1} and g^{-1} are cont.

thus $F^{-1}(c,d) = (f^{-1}(c), g^{-1}(d))$

claim: $M_1 \times M_2$ is a manifold

proof:

Hausdorff: the product of two Hausdorff spaces is Hausdorff

2^{nd} countable: " " second-countable spaces X and Y is second countable

because if U_i, V_j are bases for X, Y, $U_i \times V_j$ is a basis for $X \times Y$

I don't think you assume manifolds to be 2^{nd} countable? (only manifolds with boundary?)

locally Euclidean:

the product of two locally Euclidean spaces X and Y is Locally Euclidean

for $(x,y) \in X \times Y$ consider $U \times V$ where $U \ni x$ is a neighborhood of x hom. to $\mathbb{R}^n$ and $V \ni y$ is a nghbhood hom. to $\mathbb{R}^k$

then $U \times V$ is a neighborhood of (x,y) hom. to $\mathbb{R}^{n+k}$

$\Rightarrow$ $M_1 \times M_2$ is a manifold of dimension $n+k$ $\square$

(v)

$\boxed{2}$

claim: every topological group is Hausdorff

proof: we have the diagonal criterion:

A space X is Hausdorff $\Leftrightarrow$ Δ_X is closed

so we need to show that Δ_X is the inverse image of a closed set under a cont. map.

consider $f: (g,h) \mapsto gh^{-1}$

claim: f is continuous

proof. according to the definition of a top. group, f is a composition of continuous functions, i.e. id and $h \mapsto h^{-1}$ ✓

claim: $f(t,x) = 1$ & $f(x,y)=1 \Leftrightarrow (x,y)=(x,x)$

proof: $f(t,x) = xx^{-1} = 1$

now we must consider two cases:

case 1: $\{1\}$ is closed we assume top. groups to be T_1

then $f^{-1}\{1\} = \Delta_X$ is closed since f is continuous

case 2: $\{1\}$ is open

then the topological group is discrete, hence Hausdorff

$\square$

4

claim: $Gl_n(\mathbb{R})$ is not connected

proof:

If X is connected and $f: X \to Y$ is cont., then Y is connected
no, but $f(X)$ is connected

contraposition:

Y not connected and $f: X \to Y$ cont. $\Rightarrow$ X not connected

we take f to be the determinant function

$\det: Mat_n(\mathbb{R}) \to \mathbb{R}$

det is continuous

hence $\det^{-1}\{x \in \mathbb{R} : x < 0\}$ is open in $Mat_n(\mathbb{R})$

and $\det^{-1}\{x \in \mathbb{R} : x > 0\}$ is open in $Mat_n(\mathbb{R})$

thus we get two open subsets of $Mat_n(\mathbb{R})$

<u>claim:</u> $X = GL^+_n(\mathbb{R}) \sqcup GL^-_n(\mathbb{R})$ is not connected

to show: $\mathbb{R}^+ \cup \mathbb{R}^-$ is not connected

$\overline{\mathbb{R}^+} \cap \mathbb{R}^- = \emptyset = \mathbb{R}^+ \cap \overline{\mathbb{R}^-}$ $\square$

✓

$\boxed{5}$

<u>claim:</u> $O_n(\mathbb{R})$ is compact

proof: we show that $O_n(\mathbb{R})$ is closed and bounded

by Heine-Borel, this is equivalent to compactness

closed: consider the following map:

$$T : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$$
$$A \mapsto AA^t$$

<u>claim:</u> T is continuous

proof: every entry in the matrix AA^t is a polynomial on the entries of A $\square$

✓

now $O_n(\mathbb{R}) = T^{-1}(\{I_n\})$ where I_n is the identity matrix

$\{I_n\}$ is a closed set $\overset{T \text{ cont.}}{\Longrightarrow}$ $O_n(\mathbb{R})$ is closed

bounded: we use the hint to deduce that

each column or row is a vector of magnitude 1 ◊

ok

$\Rightarrow$ $O_n(\mathbb{R})$ is compact

✓

<u>Bonus</u>

<u>claim:</u> $O_n(\mathbb{R})$ has two connected components

<u>proof:</u> $O_n(\mathbb{R})$ is a subgroup of $Gl_n(\mathbb{R})$

$\overset{\text{Ex. 4}}{\Longrightarrow}$ $O_n(\mathbb{R})$ is not connected

$\Rightarrow$ $O_n(\mathbb{R})$ has at least two components

<u>to show:</u> $O_n(\mathbb{R})$ has not more than two components

we have $\det(A) = \pm 1$ for $A \subset O_n(\mathbb{R})$

1 and -1 are not connected

? $\{1\}$ is connected
· $\{-1\}$ is connected

det is continuous

thus $\det^{-1}(1)$ and $\det^{-1}(-1)$ are

not connected ?

1 and -1 are connected ?

thus $\det^{-1}(1)$ is connected and same for $\det^{-1}(-1)$

image connected does not imply preimage connected

$\boxed{3}$

(1) $\iota : \mathbb{R}^n \to \mathbb{RP}^n$

$(x_1, \dots, x_n) \mapsto (1 : x_1 : \dots : x_n)$

claim: ι is an embedding

def: an embedding of X into Y is a cont. map $\iota : X \hookrightarrow Y$

which yields a hom. $\iota : X \to \iota(X)$

where $\iota(X)$ is equipped with subspace top.

to show: ι is a hom. onto its image

proof: ι is clearly ? injective :

ι is continuous since its components are cont.

$(1 : x_1 : \dots : x_n)$ belongs to the set $U_1 = \{x = (1 : x_1 : \dots : x_n) : x_i \in \mathbb{R}\}$

$\iota^{-1}(U_1)$ is open in $\mathbb{R}^n$, thus U_1 is open hence v is open ? You need to show that the image of each open set is open.

$\Rightarrow$ v is a homeomorphism onto its image

(2)

Topology ex. sheet 12

1)

Claim: S^1 is a deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$

proof: consider the map

$$F(x,t) = \left((1-t) + \frac{t}{\|x\|}\right) x$$

we show that F is a deformation retraction

- S^n is a subspace of $\mathbb{R}^{n+1}$
- $f_0(x) = F(x,0) = \left((1-0) + \frac{0}{\|x\|}\right) x = x \Rightarrow f_0 = id$
- $f_1(x) = F(x,1) = \left((1-1) + \frac{1}{\|x\|}\right) x = \frac{x}{\|x\|} \in S^1$
- $f_t|_{S^1} = F(x,t)|_{S^1} = \left|(1-t) + \frac{t}{\|\frac{x}{\|x\|}\|}\right| \cdot x = \left(1-t+\frac{t}{1}\right) x = x = id$
- as a composition of cont. functions, F is continuous

Thus F is a deformation retraction ●

2)

claim: $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

proof: first, we show a "help-claim":

help-claim: $f: Z \to X \times Y$ is continuous $\Leftrightarrow$ $g: Z \to X$ and $h: Z \to Y$ defined by $f(z) = (g(z), h(z))$ are both cont.

proof "$\Rightarrow$"

let f be cont., let $V \in \mathcal{T}_X$ and $U \in \mathcal{T}_Y$

thus, from definition of product topology, $V \times U \in \mathcal{T}_{X \times Y}$

since f is cont., $f^{-1}(V \times U) := A \in \mathcal{T}_Z$

then $f_1(A) = g(A) = V \Rightarrow g^{-1}(V) = A \in \mathcal{T}_X$

similar argument for U

$\Rightarrow$ g and h are cont.

"$\Leftarrow$"

let g and h be cont., let $G \in \mathcal{T}_{X \times Y}$

by def. of prod. top. $\exists$ $V \in \mathcal{T}_X$, $U \in \mathcal{T}_Y$, both open s.t. $G := V \times U$

g cont. $\Rightarrow$ $g^{-1}(V) \in \mathcal{T}_Z$

h cont. $\Rightarrow$ $h^{-1}(U) \in \mathcal{T}_Z$

$\Rightarrow$ $\exists$ $A \in \mathcal{T}_Z$ s.t. $A := g^{-1}(V) = h^{-1}(U)$

thus f is cont. $\bullet$

now let's get back to the main proof:

let α a loop in $X \times Y$

there is a bijection Ω between such loops α in $X \times Y$ and pairs of loops (α_X, α_Y) where α_X is a loop in X, and α_Y a loop in Y

let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the projections

then the bijection Ω is given by: $\alpha \mapsto (\pi_X \circ \alpha, \pi_Y \circ \alpha)$

let $z_0 \in Z$

a map $g : Z \times I \to X \times Y$, $(z,t) \mapsto (g_X(z,t), g_Y(z,t))$

is a homotopy relative to $\{z_0\}$ iff g_X, g_Y are homotopies rel. to $\{z_0\}$

thus, loops β and γ in $X \times Y$ are homotopic iff their projections to X and Y are homotopic

hence $\beta \simeq \gamma$ iff $\pi_X \circ \beta \simeq \pi_Y \circ \gamma$ and $\pi_Y \circ \beta \simeq \pi_X \circ \gamma$

thus the bijection Ω induces a bijection σ :

$$\sigma : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$$

σ is given by the homomorphisms induced by the projections

$[\beta] \mapsto ([\pi_X \circ \beta], [\pi_Y, \beta])$

the maps induced by the projections are homomorphisms

hence the projection is a homomorphism

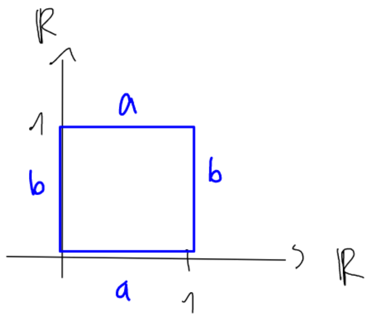
and thus an isomorphism $\bullet$

3)

claim: Consider $\mathbb{T}^2$ as the square $[-1,1]^2$ with opp. sides identified

$\mathbb{T} \setminus \{(0,0)\}$ deformation retracts onto ∞

proof: We have



the edges a and b correspond to the longitudinal and meridian circles of the torus

consider the map f on $I^2 \setminus \{0\}$ defined by

$$f: x \longmapsto \frac{x}{|x|}$$

What is f_1?

$f_1(I^2) = \partial I_2$ and $g := f_1|_{\partial I^2} = S^1$

??

thus, f is a retraction ??

and $g^{-1} \circ f$ is a retraction sending all points on $I^2 \setminus \{0\}$ to ∂I^2

now define a homotopy H by $H(x,t) = (1-t)x + t(g^{-1} \circ f)$

- $H(x,0) = x = id$
- $H(x,1) = (g^{-1} \circ f)(x) \in \partial I^2$
- $H(x,t)|_{\partial I^2} = x - tx + tx = x = id$
- H is cont. as comp. of cont. functions

∂I^2 is a graph of two circles intersecting in a point ●

4)

Brouwer fixed point theorem

(i)

claim: there is no retraction of $\bar{\mathbb{B}}^2$ onto $\partial\bar{\mathbb{B}}^2 = \mathbb{S}^1$

proof: assume we can construct a retraction $r : \bar{\mathbb{B}}^2 \to \mathbb{S}^1$

let $f : \bar{\mathbb{B}}^2 \to \bar{\mathbb{B}}^2$ have no fixed points

consider the ray $p_x(t) = (1-t) f(x) + tx$

let $r(x) = p_x(t)$ where $\|p_x(t)\| = 1$

(ii)

assume there is no fixed point for f

we use example (2) of lecture 12

we have that $f(p) = p + t_0(p-p) = p$ ↯

Why?

5)

claim: $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^m$ for $m>2$

proof: assume $\mathbb{R}^2 \cong \mathbb{R}^m$

then also $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^m \setminus \{0\}$

and $S^1 \simeq \mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^m \setminus \{0\} \simeq S^{m-1}$

$$\Rightarrow \quad \pi_1(S^1) \cong \pi_1(S^{m-1})$$

but $\pi_1(S^1) \cong \mathbb{Z}$ $\quad \Rightarrow \quad 2 = m$

why?? –